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ON GENERALIZED TWO-DIMENSIONAL PLATE THEORY—I

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Abstract—We derive two-dimensional equations for transverse bending of plates including the effect of moment stresses. The method of derivation is analogous to what has earlier been done without moment stress, using a suitable version of the variational principle for stresses *and* displacements. The two-dimensional theory of this paper is of the twelfth order and contains as limiting cases known sixth order theories in which moment stresses are either negligible or dominant, as well as a fourth-order theory involving both force and moment stresses, which applies when transverse shear deformation is negligible.

1. INTRODUCTION

In what follows, we consider the derivation of two-dimensional plate theory from three-dimensional elasticity theory for the case of a medium supporting moment stresses as well as force stresses, by means of the same variational approach which has been used some time ago [3, 4] for the derivation of plate theory including the effect of transverse shear deformation, from "ordinary" elasticity theory. The principal purpose of the present considerations is the gaining of additional insight in connection with a procedure for deriving two-dimensional results by iteration from an integro-differential equation formulation of the three-dimensional plate and shell problem which has recently been proposed [5].

The problem of deriving two-dimensional plate theories on the basis of three-dimensional moment stress elasticity theory has previously been considered by Eringen [1] and by Green and Naghdi [2]. The latter use asymptotic methods for the derivation of rational results for a special class of problems. The former uses certain averages of the equations of the three-dimensional problem for the derivation of a semi-rational two-dimensional theory. The present variational approach may subsequently be interpreted as an averaging procedure with automatic rationality of the resultant two-dimensional theory.

2. THE BOUNDARY VALUE PROBLEM IN VECTORIAL FORM

We consider a region in space bounded by a cylindrical surface $f(x, y) = 0$ and by two parallel planes $z = \pm c$. We assume the existence of force stress vectors $\boldsymbol{\sigma}_x, \boldsymbol{\sigma}_y, \boldsymbol{\sigma}_z$ and moment stress vectors $\boldsymbol{\tau}_x, \boldsymbol{\tau}_y, \boldsymbol{\tau}_z$ and, with force and moment load intensity vectors $\mathbf{p}$ and $\mathbf{q}$, the validity of equilibrium equations of the form

$$\boldsymbol{\sigma}_{x,x}+\ldots+\mathbf{p}=0, \qquad \boldsymbol{\tau}_{x,x}+\ldots+\mathbf{i}\times\boldsymbol{\sigma}_x+\ldots+\mathbf{q}=0. \tag{1}$$

Considerations of virtual work then lead to relations between displacement vectors $\mathbf{v}$ and $\boldsymbol{\psi}$ and force and moment strain vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ and $\mathbf{k}_x, \mathbf{k}_y, \mathbf{k}_z$ of the form

$$\mathbf{e}_x = \mathbf{v}_{,x}+\mathbf{i}\times\boldsymbol{\psi},\ldots, \qquad \mathbf{k}_x = \boldsymbol{\psi}_{,x},\ldots. \tag{2}$$

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The above differential equations are supplemented by a system of relations which express the property of elasticity of the medium, in the symbolic form

$$\mathbf{e}_x = \partial B/\partial \boldsymbol{\sigma}_x, \ldots, \qquad \mathbf{k}_x = \partial B/\partial \boldsymbol{\tau}_x, \ldots \tag{3}$$

where B is a given function of the indicated arguments.

Among various possible systems of boundary conditions we here consider the case of stress boundary conditions, in the form

$$\boldsymbol{\sigma}_n = \bar{\boldsymbol{\sigma}}_n, \qquad \boldsymbol{\tau}_n = \bar{\boldsymbol{\tau}}_n \tag{4}$$

where the subscript n indicates the outward normal direction on the plane boundary portions $z =$ const. as well as on the cylindrical boundary portion $f = 0$.

Equations (1) to (4) are associated with a variational equation for stresses and displacements, $\delta I = 0$, [4, 6] where

$$I = \int [(\mathbf{v}_{,x}+\mathbf{i}\times\boldsymbol{\psi})\cdot\boldsymbol{\sigma}_x+\ldots+\boldsymbol{\psi}_{,z}\cdot\boldsymbol{\tau}_z-B-\mathbf{v}\cdot\mathbf{p}-\boldsymbol{\psi}\cdot\mathbf{q}]\,\mathrm{d}V-\oint[\mathbf{v}\cdot\bar{\boldsymbol{\sigma}}_n+\boldsymbol{\psi}\cdot\bar{\boldsymbol{\tau}}_n]\,\mathrm{d}S. \tag{5}$$

The Euler equations associated with (5) are equations (1) and (4) together with (2) or (3), depending on whether equations (2) or equations (3) are considered as equations of definition for $\mathbf{e}_x$, $\mathbf{k}_x$, etc.

3. THE BOUNDARY VALUE PROBLEM IN SCALAR FORM

We write

$$(\boldsymbol{\sigma}_x, \mathbf{e}_x, \mathbf{v}) = (\sigma_{xx}, e_{xx}, v_x)\mathbf{i}+(\sigma_{xy}, e_{xy}, v_y)\mathbf{j}+(\sigma_{xz}, e_{xz}, v_z)\mathbf{k} \tag{6}$$

etc. and, for convenience sake, introduce a preferential treatment of the z-direction in the expressions for $\boldsymbol{\tau}$, $\mathbf{k}$ and $\boldsymbol{\psi}$ by writing

$$(\boldsymbol{\tau}_x, \mathbf{k}_x, \boldsymbol{\psi}) = \mathbf{k}\times[(\tau_{xx}, k_{xx}, \psi_x)\mathbf{i}+\ldots]+(\tau_{xz}, k_{xz}, \psi_z)\mathbf{k} \tag{7}$$

etc. Assuming for simplicity the absence of $\mathbf{p}$ and $\mathbf{q}$, the equilibrium equations (1) assume then the form

$$\begin{aligned}
\sigma_{xx,x}+\sigma_{yx,y}+\sigma_{zx,z} &= 0, & \tau_{xx,x}+\tau_{yx,y}+\tau_{zx,z}+\sigma_{zx}-\sigma_{xz} &= 0\\
\sigma_{xy,x}+\sigma_{yy,y}+\sigma_{zy,z} &= 0, & \tau_{xy,x}+\tau_{yy,y}+\tau_{zy,z}+\sigma_{zy}-\sigma_{yz} &= 0\\
\sigma_{xz,x}+\sigma_{yz,y}+\sigma_{zz,z} &= 0, & \tau_{xz,x}+\tau_{yz,y}+\tau_{zz,z}+\sigma_{xy}-\sigma_{yx} &= 0
\end{aligned} \tag{8}$$

and the strain displacement relations (2) become

$$\begin{aligned}
&e_{xx} = v_{x,x}, \quad e_{xy} = v_{y,x}-\psi_z, \quad e_{yx} = v_{x,y}+\psi_z, \quad e_{xz} = v_{z,x}+\psi_x, \ldots,\\
&k_{xx} = \psi_{x,x}, \quad k_{xy} = \psi_{y,x}, \ldots.
\end{aligned} \tag{9}$$

Stress–strain relations corresponding to the symbolic equations (3) are in terms of the above components of stress and strain

$$e_{xx} = \partial B/\partial\sigma_{xx}, \qquad e_{xy} = \partial B/\partial\sigma_{xy}, \ldots, \qquad k_{xx} = \partial B/\partial\tau_{xx}, \ldots. \tag{10}$$

In the statement of scalar boundary conditions we assume for simplicity that the plane boundary portions are free of stress, i.e.

$$z = \pm c; \qquad \sigma_{zx} = \sigma_{zy} = \sigma_{zz} = \tau_{zx} = \tau_{zy} = \tau_{zz} = 0 \tag{11}$$

and we write the scalar conditions corresponding to (4) for the cylindrical boundary portions in the form

$$f = 0; \qquad \sigma_{nn} = \bar{\sigma}_{nn}, \qquad \sigma_{ns} = \bar{\sigma}_{ns}, \qquad \sigma_{nz} = \bar{\sigma}_{nz}, \qquad \tau_{nn} = \bar{\tau}_{nn}, \ldots \tag{12}$$

With this the variational equation (5) becomes

$$\delta\Big\{\iiint[v_{x,x}\sigma_{xx}+(v_{y,x}-\psi_z)\sigma_{xy}+(v_{x,y}+\psi_z)\sigma_{yx}+\ldots$$
$$+(v_{z,x}+\psi_x)\sigma_{xz}+\ldots+\psi_{x,x}\tau_{xx}+\psi_{y,x}\tau_{xy}+\ldots-B]\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y \tag{13}$$
$$-\oint\int[\bar{\sigma}_{nn}v_n+\bar{\sigma}_{ns}v_s+\bar{\sigma}_{nz}v_z+\bar{\tau}_{nn}\psi_n+\ldots]\,\mathrm{d}z\,\mathrm{d}s\Big\} = 0.$$

4. A CLASS OF STRESS–STRAIN RELATIONS

In what follows consideration is restricted to the class of transversely isotropic linear media for which the strain potential function B is of the form

$$\begin{aligned}B = {}& \frac{\sigma_{xx}^2+\sigma_{yy}^2-2\nu_\sigma\sigma_{xx}\sigma_{yy}+(1+\nu_\sigma)(\sigma_{xy}^2+\sigma_{yz}^2)+\mu_\sigma(\sigma_{xy}-\sigma_{yz})^2}{2E}\\ &+\frac{\sigma_{zz}^2}{2E_z}-\nu_{\sigma z}\frac{(\sigma_{xx}+\sigma_{yy})\sigma_{zz}}{2E}+\frac{(\sigma_{zx}+\sigma_{xz})^2+(\sigma_{zy}+\sigma_{yz})^2}{8G}\\ &+\frac{(\sigma_{zx}-\sigma_{xz})^2+(\sigma_{zy}-\sigma_{yz})^2}{8G^*}+\frac{\tau_{xz}^2+\tau_{yz}^2}{2c^2\Lambda}+\frac{\tau_{zx}^2+\tau_{zy}^2}{2c^2\Lambda_z}+\frac{\tau_{zz}^2}{2c^2\Gamma_z}\\ &+\frac{\tau_{xx}^2+\tau_{yy}^2-2\nu_z\tau_{xx}\tau_{yy}+(1+\nu_\tau)(\tau_{xy}^2+\tau_{yx}^2)+\mu_\tau(\tau_{yx}-\tau_{xy})^2}{2c^2\Gamma}.\end{aligned} \tag{14}$$

Additionally, the given functions of position E, ν, etc. are assumed to be constants.†

5. A CLASS OF STRESS STATES FOR TRANSVERSE BENDING OF PLATES

In generalization of corresponding assumptions in the absence of moment stresses we take as expressions for the components of force and moment stresses which are responsible for plate bending and twisting moments M_{xx}, M_{xy}, M_{yx}, M_{yy},

$$\sigma_{xx} = \frac{3M_{xx}^\sigma}{2c^2}\frac{z}{c}, \qquad \sigma_{xy} = \frac{3M_{xy}^\sigma}{2c^2}\frac{z}{c}, \ldots, \qquad \tau_{xx} = \frac{M_{xx}^\tau}{2c}, \ldots \tag{15}$$

whereupon $M_{xx} = M_{xx}^\sigma + M_{xx}^\tau$, etc.

† Also omitted from consideration, for simplicity, are terms of the form $\sigma_{xz}\sigma_{zx}+\sigma_{yz}\sigma_{zy}$ and terms of the form $\sigma\tau$ which would be compatible with the assumptions of linearity and transverse isotropy.

We next choose as expressions for shear stresses acting between midplane parallel layers, as in the theory without moment stresses,

$$\sigma_{zx} = \frac{3Q_x^\sigma}{4c}\left(1-\frac{z^2}{c^2}\right), \qquad \sigma_{zy} = \frac{3Q_y^\sigma}{4c}\left(1-\frac{z^2}{c^2}\right) \tag{16}$$

where Q_x^σ and Q_y^σ are as yet undetermined functions of x and y.

Inspection of the first two moment equilibrium equations in (8) indicates that it will be good to choose as expressions for shear stresses acting between midsurface perpendicular layers

$$\sigma_{xz} = \frac{3Q_x^\sigma}{4c}\left(1-\frac{z^2}{c^2}\right)+\frac{Q_x^\tau}{2c}, \qquad \sigma_{yz} = \frac{3Q_y^\sigma}{4c}\left(1-\frac{z^2}{c^2}\right)+\frac{Q_y^\tau}{2c}. \tag{17}$$

With this we will have transverse shear stress resultants Q of the form $Q = Q^\sigma + Q^\tau$.

Inspection of the third force equilibrium equation in (8), in conjunction with (17) and (11), suggests as expression for the remaining component of transverse normal stress

$$\sigma_{zz} = \frac{S}{4}\left(\frac{z}{c}-\frac{z^3}{c^3}\right). \tag{18}$$

It remains to choose expressions for five of the moment stress components. Consideration of the three moment equations in (8) and of the face boundary conditions (11), in conjunction with the assumptions (15) to (17), suggests as simple reasonable assumptions

$$\tau_{zx} = \tau_{zy} = \tau_{xz} = \tau_{yz} = 0, \qquad \tau_{zz} = \frac{3T}{4c}\left(1-\frac{z^2}{c^2}\right). \tag{19}†$$

Having equations (15)–(19) we must next establish a system of two-dimensional differential equations and boundary conditions for the quantities M, Q, S and T, together with a suitably defined set of displacement measures, in such a way that equilibrium conditions as well as stress–strain relations are satisfied in a systematically approximate fashion.

6. VARIATIONAL DERIVATION OF TWO-DIMENSIONAL DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS

Introduction of equations (15)–(19) for stresses into the variational equation (13) leads to the definition of two-dimensional displacement measures α, β, w^*, w and Ω as follows:

$$(\alpha_x, \alpha_y) = \frac{3}{2c^3}\int_{-c}^{c}(v_x, v_y)z\,dz, \qquad (\beta_x, \beta_y) = \frac{1}{2c^2}\int_{-c}^{c}(\psi_x, \psi_y)\,dz \tag{20}$$

$$\Omega = \frac{3}{2c^3}\int_{-c}^{c}\psi_z z\,dz, \qquad (w^*, w) = \int_{-c}^{c}\left[\frac{3}{4c}\left(1-\frac{z^2}{c^2}\right), \frac{1}{2c}\right]v_z\,dz. \tag{21}$$

† To indicate the nature of the considerations involved in the foregoing we note that a refinement of (19) would be to choose $\tau_{xz} = (z/c)T_x$, $\tau_{zx} = [(z/c)-(z/c)^3]T_{zx}$, etc. while at the same time replacing Q_x^σ and Q_y^σ in (16) (but not in (17)) by quantities Q_{zx} and Q_{zy}.

Therewith we have

$$\iiint [v_{x,x}\sigma_{xx}+(v_{y,x}-\psi_z)\sigma_{xy}+\ldots+\psi_{z,z}\tau_{zz}]\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$$

$$=\iint [\alpha_{x,x}M^{\sigma}_{xx}+(\alpha_{y,x}-\Omega)M^{\sigma}_{xy}+(\alpha_{x,y}+\Omega)M^{\sigma}_{yx}+\ldots$$
$$+(\alpha_x+w^*_{,x})Q^{\sigma}_x+\ldots+\beta_{x,x}M^{\tau}_{xx}+\beta_{y,x}M^{\tau}_{xy}+\ldots$$
$$+(\beta_x+w_{,x})Q^{\tau}_x+\ldots+(w-w^*)S+\Omega T]\,\mathrm{d}x\,\mathrm{d}y. \tag{22}$$

Furthermore, with B given by (14),

$$\iiint B\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y=\iint\Big\{\frac{1}{2D^*_\sigma}[(M^{\sigma}_{xx})^2+(M^{\sigma}_{yy})^2-2\nu_\sigma M^{\sigma}_{xx}M^{\sigma}_{yy}$$
$$+(1+\nu_\sigma)\,((M^{\sigma}_{xy})^2+(M^{\sigma}_{yx})^2)+\mu_\sigma(M^{\sigma}_{xy}-M^{\sigma}_{yx})^2]$$
$$+\frac{cS^2}{210E_z}-\nu_{\sigma z}\frac{(M^{\sigma}_{xx}+M^{\sigma}_{yy})S}{10Ec}$$
$$+\frac{1}{2D^*_\tau}[(M^{\tau}_{xx})^2+\ldots+\mu_\tau(M^{\tau}_{xy}-M^{\tau}_{yx})^2]$$
$$+\frac{1}{Gc}\left[\frac{3(Q^{\sigma}_x)^2}{10}+\frac{Q^{\sigma}_xQ^{\tau}_x}{4}+\frac{(Q^{\tau}_x)^2}{16}\right]$$
$$+\frac{(Q^{\tau}_x)^2}{16G^*c}+\ \ldots\ +\frac{3T^2}{10\Gamma_z c^3}\Big\}\,\mathrm{d}x\,\mathrm{d}y \tag{23}$$

where $D^*_\sigma=\frac{2}{3}Ec^3$ and $D^*_\tau=2\Gamma c^3$.

The remaining double integral in (13) reduces, with the help of expressions for edge stresses consistent with equations (15), (17) and (19), to a single integral, as follows:

$$\oint\int[\bar{\sigma}_{nn}v_n+\ldots+\bar{\tau}_{nz}\psi_z]\,\mathrm{d}z\,\mathrm{d}s$$
$$=\oint[\overline{M}^{\sigma}_{nn}\alpha_n+\overline{M}^{\sigma}_{ns}\alpha_s+\overline{Q}^{\sigma}_n w^*+\overline{Q}^{\tau}_n w+\overline{M}^{\tau}_{nn}\beta_n+\overline{M}^{\tau}_{ns}\beta_s]\,\mathrm{d}s. \tag{24}$$

Having equations (22) to (24), with all indicated scalar stress and displacement variables being varied independently, we now obtain systems of Euler differential equations and boundary conditions consisting of seven equilibrium equations

$$M^{\sigma}_{xx,x}+M^{\sigma}_{yx,y}-Q^{\sigma}_x=0,\qquad M^{\sigma}_{xy,x}+M^{\sigma}_{yy,y}-Q^{\sigma}_y=0 \tag{25}$$

$$Q^{\sigma}_{x,x}+Q^{\sigma}_{y,y}+S=0,\qquad Q^{\tau}_{x,x}+Q^{\tau}_{y,y}-S=0 \tag{26}$$

$$M^{\tau}_{xx,x}+M^{\tau}_{yx,y}-Q^{\tau}_x=0,\qquad M^{\tau}_{xy,x}+M^{\tau}_{yy,y}-Q^{\tau}_y=0 \tag{27}$$

$$M^{\sigma}_{yx}-M^{\sigma}_{xy}+T=0 \tag{28}$$

together with fourteen stress–strain relations

$$\alpha_{x,x} = \frac{M^{\sigma}_{xx} - \nu_{\sigma} M^{\sigma}_{yy}}{D^*_{\sigma}} - \frac{\nu_{\sigma z} S}{10Ec}, \qquad \alpha_{y,y} = \frac{M^{\sigma}_{yy} - \nu_{\sigma} M^{\sigma}_{xx}}{D^*_{\sigma}} - \frac{\nu_{\sigma z} S}{10Ec} \tag{29}$$

$$D^*_{\sigma}(\alpha_{y,x} - \Omega) = (1+\nu_{\sigma})M^{\sigma}_{xy} + \mu_{\sigma}(M^{\sigma}_{xy} - M^{\sigma}_{yx}) \tag{30}$$

$$D^*_{\sigma}(\alpha_{x,y} + \Omega) = (1+\nu_{\sigma})M^{\sigma}_{yx} + \mu_{\sigma}(M^{\sigma}_{yx} - M^{\sigma}_{xy}) \tag{31}$$

$$\alpha_x + w^*_{,x} = \frac{3Q^{\sigma}_x}{5Gc} + \frac{Q^{\tau}_x}{4Gc}, \qquad \alpha_y + w^*_{,y} = \frac{3Q^{\sigma}_y}{5Gc} + \frac{Q^{\tau}_y}{4Gc} \tag{32}$$

$$D^*_{\tau}\beta_{x,x} = M^{\tau}_{xx} - \nu_{\tau} M^{\tau}_{yy}, \qquad D^*_{\tau}\beta_{y,y} = M^{\tau}_{yy} - \nu_{\tau} M^{\tau}_{xx} \tag{33}$$

$$D^*_{\tau}\beta_{y,x} = (1+\nu_{\tau})M^{\tau}_{xy} + \mu_{\tau}(M^{\tau}_{xy} - M^{\tau}_{yx}) \tag{34}$$

$$D^*_{\tau}\beta_{x,y} = (1+\nu_{\tau})M^{\tau}_{yx} + \mu_{\tau}(M^{\tau}_{yx} - M^{\tau}_{xy}) \tag{35}$$

$$\beta_x + w_{,x} = \left(\frac{1}{G} + \frac{y}{G^*}\right)\frac{Q^{\tau}_x}{8c} + \frac{Q^{\sigma}_x}{yGc}, \qquad \beta_y + w_{,y} = \left(\frac{1}{G} + \frac{1}{G^*}\right)\frac{Q^{\tau}_y}{8c} + \frac{Q^{\sigma}_y}{4Gc} \tag{36}$$

$$w - w^* = \frac{cS}{105E_z} - \nu_{\sigma z}\frac{M^{\sigma}_{xx} + M^{\sigma}_{yy}}{10Ec}, \qquad \Omega = \frac{3T}{5c^3\Gamma_z} \tag{37}$$

and of six (stress) boundary conditions

$$\Delta M^{\sigma}_{nn} = \Delta M^{\sigma}_{ns} = \Delta Q^{\sigma}_{n} = \Delta Q^{\tau}_{n} = \Delta M^{\tau}_{nn} = \Delta M^{\tau}_{ns} = 0 \tag{38}$$

along $f = 0$, where $\Delta M^{\sigma}_{nn} \equiv M^{\sigma}_{nn} - \overline{M}^{\sigma}_{nn}$, etc.†

The system (25) to (37), in its generality, may readily be reduced to a system of seven equations for seven displacement variables α_x, α_y, β_x, β_y, w, w^*, Ω, by introducing the stress–strain relations (29) to (37) into the seven equilibrium equations (25) to (28). This reduction will not be carried out here except for the following part of it. We use the equilibrium equation (28) together with the stress–strain relations (30), (31) and (37) to express Ω in terms of α_x and α_y, in the form

$$\Omega = \frac{1}{2}\,\frac{\alpha_{y,x} - \alpha_{x,y}}{1 + \frac{5}{4}(1+\nu_{\sigma}+2\mu_{\sigma})\Gamma_z/E}. \tag{39}$$

Therewith, the stress–strain relation (30) may be written in the Ω-free form

$$M^{\sigma}_{xy} = \frac{D^*_{\sigma}}{1+\nu_{\sigma}}\left\{\alpha_{y,x} - \frac{1+\frac{5}{2}\mu_{\sigma}\Gamma_z/E}{1+\frac{5}{4}(1+\nu_{\sigma}+2\mu_{\sigma})\Gamma_z/E}\,\frac{\alpha_{y,x} - \alpha_{x,y}}{2}\right\} \tag{40}$$

with M^{σ}_{yx} given by an analogous expression.

Of particular significance in equations (39) and (40) is the special case

$$\Gamma_z = 0; \qquad \Omega = \frac{\alpha_{y,x} - \alpha_{x,y}}{2}, \qquad \frac{M^{\sigma}{}_{xy}}{D} = \frac{M^{\sigma}_{yx}}{D} = \frac{\alpha_{y,x} + \alpha_{x,y}}{2(1+\nu_{\sigma})}. \tag{41}$$

Plate theory with no transverse shear and normal strains. Setting

$$\frac{1}{G} = \frac{1}{G^*} = \frac{1}{E_z} = \nu_{\sigma z} = 0 \tag{42}$$

† The occurrence of separate boundary conditions for M^{σ}_{nn}, M^{σ}_{ns}, M^{τ}_{nn}, M^{τ}_{ns}, in a general, two-dimensional plate theory including moment stresses, had been suggested to the author by A. E. Green, and P. M. Naghdi prior to the work contained in this article.

we have from (32), (36) and (37)

$$w^* = w, \qquad \alpha_x = \beta_x = -w_{,x}, \qquad \alpha_y = \beta_y = -w_{,y}. \tag{43}$$

Equations (29) and (40) give, with $D_\sigma = D_\sigma^*/(1-\nu_\sigma^2)$,

$$M_{xx}^\sigma = -D_\sigma(w_{,xx}+\nu_\sigma w_{,yy}), \qquad M_{xy}^\sigma = -(1-\nu_\sigma)D_\sigma w_{,xy}, \qquad \text{etc.,} \tag{44}$$

while (33) to (35) imply, with $D_\tau = D_\tau^*/(1-\nu_\tau^2)$

$$M_{xx}^\tau = -D_\tau(w_{,xx}+\nu_\tau w_{,yy}), \qquad M_{xy}^\tau = -(1-\nu_\tau)D_\tau w_{,xy}, \qquad \text{etc.} \tag{45}$$

Introduction of $M_{xx} = M_{xx}^\sigma + M_{xx}^\tau$, etc. into the relation $M_{xx,xx}+2M_{xy,xy}+M_{yy,yy} = 0$, which may be deduced from (25) to (27), results in a fourth-order equation for w, which for the case of constant D_σ, D_τ, ν_σ, ν_τ assumes the form $\nabla^2\nabla^2 w = 0$, the same as in the theory without moment stresses. Again as in the theory without moment stresses, the kinematical constraints (32) result in a contraction of the boundary conditions (38) to the classical form $\Delta M_{nn} = \Delta Q_n + (\Delta M_{ns})_{,s} = 0$.

7. SIXTH-ORDER PLATE THEORIES

Setting $D_\tau^* = 0$ we have from (33) to (35) that $M_{xx}^\tau = M_{yy}^\tau = M_{xy}^\tau = M_{yx}^\tau = 0$, and then from (27) $Q_x^\tau = Q_y^\tau = 0$, and from (26) $S = 0$. The remaining system of equations for α_x, α_y, w^* is equivalent to the previously given results for plate theory including transverse shear deformation [3].

Setting $D_\sigma^* = 0$, and therewith $M_{xx}^\sigma = M_{yy}^\sigma = M_{xy}^\sigma = M_{yx}^\sigma = Q_x^\sigma = Q_y^\sigma = S = T = 0$, the remaining system of equations for β_x, β_y, w is equivalent to the system which Green and Naghdi obtain by asymptotic considerations [2].

Effectively, the former theory will be applicable as long as $D_\tau^* \ll D_\sigma^*$ and the latter theory will be applicable as long as $D_\sigma^* \ll D_\tau^*$.

It may be seen as follows that when D_τ^* and D_σ^* are of the same order of magnitude that then, in general, the complete twelfth order system must be used. We first write equations (32) and (36) with the help of (25), (27) and (37) in the form

$$\alpha_x + w_{,x}^* = \frac{3}{5}\frac{M_{xx,x}^\sigma + M_{yx,y}^\sigma}{Gc} + \frac{M_{xx,x}^\tau + M_{yx,y}^\tau}{4Gc} \tag{46}$$

$$\beta_x + w_{,x} = \left(1+\frac{G}{G^*}\right)\frac{M_{xx,x}^\tau + M_{yx,y}^\tau}{8Gc} + \frac{M_{xx,x}^\sigma + M_{yx,y}^\sigma}{4Gc} - \left(\frac{cS}{105E_z} - \nu_{\sigma z}\frac{M_{xx}^\sigma + M_{yy}^\sigma}{10Ec}\right)_{,x} \tag{47}$$

with analogous expressions for $\alpha_y + w_{,y}^*$ and $\beta_y + w_{,y}$. We next introduce M_{xx}^σ, M_{xx}^τ, etc. from (30), (31), (33), (34), (35), and S from (26), into these relations. From this we see that when D_σ^* and D_τ^* are of comparable magnitude then as long as *any* of the terms on the right of (46) and (47) are significant, *all* (except those with S) must be considered, and α_x and β_x remain as essentially separate dependent variables.

Retention of the terms on the right of (46) and (47) implies a characteristic length L such that $D/GcL^2 = O(1)$ or $L = O[c\sqrt{(E/G)}]$. From this it follows that when $\sqrt{(E/G)} = O(1)$ and when the smallest characteristic length of interest is to be large compared to c then *all* terms on the right of (46) and (47) are small of higher order and the simple fourth order theory of equations (43)–(45) applies.

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Абстракт—Выводятся двухмерные уравнения для поперечного изгиба пластинок включая эффект моментных напряжений. Метод вывода аналогичен сделанному ранее без учета моментных напряжений, используя удобный способ вариационного принципа для напряжений и перемещений. Двухмерная теория, представленная в настоящей работе, является двенадцатого порядка и заключает, в качестве предельных случаев, известные теории шестого порядка, в которых моментные напряжения оказываются незначительными или преобладающими, а также теорию четвертого порядка, вызывающей напряжения так от сил как и от моментов. Эта последняя теория применяется в случае, когда поперечная деформация сдвига незначительна.